Isochronous manifolds in self-triggered control

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***Abstract*— Feedback control laws are predominantly implemented on digital platforms as periodic tasks. Although periodicity simplifies the analysis and design of the implementation, new applications call for more efficient utilization of available resources such as processor utilization. To address this issue, new implementation paradigms, such as event-triggered and self-triggered control, have recently been proposed. These policies allow for a dynamic allocation of resources, since the execution times of the control task are defined according to the current state of the system. In this paper we continue our exploratory journey in the field of self-triggered control for nonlinear systems. In our previous work, we exploited the geometry of homogeneous and polynomial control systems to identify one dimensional manifolds along which the execution times scaled in a predictable manner. In this paper we complement our previous work by focusing on manifolds where the times remain constant. By merging both ideas new self-trigger conditions are derived, outperforming existing techniques.**

I. Introduction

Feedback control laws are traditionally implemented in a periodic fashion. While this approach facilitates both the design and the analysis of the implemented controller, it leads to a conservative usage of resources. With the onset of embedded systems, event-triggered control has received an increasing attention as a way to reduce the amount of resources (computations, bandwidth) required for control tasks ([ÅB02], [Årz99], [Tab07]). Under such strategy, the plant is continuously monitored to ascertain the next time at which the controller needs to be executed and the actuator updated. One of the issues that hampers the implementation of event-triggered control is the need of dedicated hardware in order to track the current state. Self-triggered control was proposed in [VFM03] as a strategy to overcome this disadvantage. In every closed-loop system, the state of the plant has to be measured or estimated to compute the control law. Thus, this information can be utilised to determine the next execution time. Self-triggered control can be considered as a way to emulate event-triggered control without resorting to extra hardware.

Self-triggered control for linear systems has recently been studied in [WL09]. Drawing inspiration from event-triggered control, state-dependent triggering conditions were developed that guarantee stability and a desired level of performance. Stabilizing self-trigger conditions for nonlinear systems were proposed in [AT08b], where the authors studied the properties of the trajectories for two classes of nonlinear systems: state-dependent homogeneous systems and polynomial systems. Such properties were exploited to derive scaling laws describing the evolution of the execution times for the control task along one-dimensional submanifolds of the state space. These manifolds, termed homogeneous rays, consist of rays radiating away from the origin and suggest a two-step strategy to determine the execution times. First, a lower bound for the execution times, valid on a ball around the origin, is computed. Second, this time is extended to the whole operating region using the aforementioned scaling law. This is achieved by applying the scaling law to the ray passing through the point whose time is to be determined. While the scaling law describes *exactly* how the times scale along homogeneous rays, the lower bound derived in the first step represents a rough estimate of the execution times.

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To improve the previously described two-step approach, we study in this paper *isochronous* manifolds. These are submanifolds of the state space containing the states for which the execution times remain constant. Replacing the ball used in the first step with an isochronous manifold allows us to compute the execution times in an *exact* manner, thereby eliminating all the sources of conservativeness in self-triggered control. In this paper, we report the following results on this research agenda:

- we prove the existence of isochronous manifolds for homogeneous and polynomial control systems;
- we provide sufficient conditions for the existence of computable tight approximations to isochronous manifolds.

We illustrate the proposed results on several examples, showing that the times computed by the proposed technique barely differ from the resulting from event-triggered techniques.

II. Notation

We shall use the notation $|x|$ to denote the Euclidean norm of an element $x \in \mathbb{R}^n$. A continuous function $\alpha : [0, a[\to \mathbb{R}_0^+$, $a > 0$, is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A function is said to be of class C^∞ or smooth if it can be differentiated infinitely many times. All the objects in this paper are considered to be smooth unless otherwise stated.

Given vector fields X and Y in an n-dimensional manifold M, we let [X,Y] denote their Lie product which, in local coordinates $x = (x_1, x_2, ..., x_n)$, we take as $\frac{\partial Y}{\partial x}(x)X(x) - \frac{\partial X}{\partial x}(x)Y(x)$. We use $(\mathcal{L}_X h)(x)$ to denote the Lie derivative of a map $h : M \to \mathbb{R}$ at a point x along the flow of the vector field $X : M \to TM$ which, in local coordinates, we take as $\frac{\partial h}{\partial x}X(x)$. Likewise, $\mathcal{L}_X^k h$ represents the kth Lie derivative, defined by $\mathcal{L}_X^0 h = h$ and $\mathcal{L}_X^k h = \mathcal{L}_X(\mathcal{L}_X^{k-1} h)$).

We consider control systems of the form:

$$\dot{\mathrm{x}} = f(\mathrm{x}, \mathrm{u}), \qquad \mathrm{x}(t) \in \mathbb{R}^n,\ \mathrm{u}(t) \in \mathbb{R}^m, t \in \mathbb{R}_0^+. \quad \text{(II.1)}$$

We denote by $x \in \mathbb{R}^n$ the state of the control system, by x a solution of (II.1), and by u the input trajectory.

Finally, we use $\mathbf{e}^s$ to represent the exponential of $s \in \mathbb{R}$, and A_{ij} to denote the element ij of the matrix A.

III. Self-triggered stabilization of nonlinear systems

We start by analyzing the behaviour of the plant under the event-triggered implementation introduced in [Tab07]. Consider the control system described in (II.1) for which a feedback controller $u = k(x)$ has been designed. In a digital implementation, this controller is executed at discrete instants of time t_i. We are interested in defining a sequence of execution times t_i for the implemented control system that guarantees stability and desired performance, while saving resources.

To proceed, we formalize the mismatch between the current state and the sampled state through the following measurement error:

$$\mathrm{e}(t) = \mathrm{x}(t_i) - \mathrm{x}(t) \qquad \text{for } t \in [t_i, t_{i+1}[\tag{III.1}$$

With this definition, the closed-loop system becomes:

$$\dot{\mathrm{x}} = f(\mathrm{x}, k(\mathrm{x}(t_i))) = f(\mathrm{x}, k(\mathrm{x}+\mathrm{e})) \tag{III.2}$$

Assume now that the control law $u = k(x)$ renders the system (III.2) input-to-state stable (see [Son05] for details) with respect to the measurement error e. Under this assumption, there exists a Lyapunov function V for the system that satisfies the following inequality:

$$\dot{V} \leq -\alpha(|x|) + \gamma(|e|) \tag{III.3}$$

where α and γ are $\mathcal{K}_\infty$ functions. It was shown in [Tab07] that stability of the closed loop system (III.2) can be guaranteed if the error is restricted to satisfy:

$$b|e| \leq \sigma a|x| \tag{III.4}$$

for a and b appropriately chosen according to the Lipschitz constants of α^{-1} and γ. Inequality (III.4) can be enforced by executing the control task whenever:

$$|e| = \sigma\frac{a}{b}|x| \tag{III.5}$$

since execution of the control task resets e to zero, *i.e.*, $\mathrm{e}(t_i) = \mathrm{x}(t_i) - \mathrm{x}(t_i) = 0$. Our goal is to find a self-trigger condition that guarantees inequality (III.4) at any instant of time, i.e, a simple formula that renders a sequence of stabilizing execution times. The execution time implicitly defined by (III.5) is the time it takes for $\frac{|\mathrm{e}|}{|\mathrm{x}|}$ to evolve from 0 to $\sigma\frac{a}{b}$. We denote this time by $\tau(\mathrm{x}(t_i))$, as it depends on the last sample of the system $\mathrm{x}(t_i)$:

$$\tau(\mathrm{x}(t_i)) := \min\{t > t_i : |\mathrm{e}(t, \mathrm{x}(t_i))| = \sigma\frac{a}{b}|\mathrm{x}(t, \mathrm{x}(t_i))|\} \tag{III.6}$$

Since the expressions of x and e are in general not known for nonlinear systems, it is not possible to compute $\tau(\mathrm{x}(t_i))$ in closed form. Hence, we exploit the geometry of homogeneous systems to avoid computing (III.6) using the flows. In the next section we revisit previous results on the evolution on the execution times τ for different classes of nonlinear systems

IV. Scaling laws for nonlinear systems

In this section we review results from [AT08b]. We start by defining the classes of systems that will be considered in this paper. Self-trigger conditions for general nonlinear systems are discussed in section VIII.

A. State-dependent homogeneous systems

Homogeneous vector fields appear as local approximations for general nonlinear vector fields [Her91] since an analytic function can always be decomposed in an infinite sum of homogeneous functions. We first define homogeneity for functions.

Definition 4.1: A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called homogeneous of degree d if for all $\lambda > 0$, there exist $r_i > 0$, $i = 1 \ldots n$ such that:

$$f_i(\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n) = \lambda^d\lambda^{r_i}f_i(x_1, \ldots, x_n) \tag{IV.1}$$

where $d > -\min_i r_i$.

Drawing inspiration from [Kaw95], the following notion of homogeneity for vector fields was introduced in [AT08b].

Definition 4.2: Let $D : M \to TM$ be a (dilation) vector field such that $\dot{\mathrm{x}} = -D(\mathrm{x})$ is globally asymptotically stable. A vector field $X : M \to TM$ is called homogeneous with degree function $\xi : M \to \mathbb{R}$ with respect to the vector field $D : M \to TM$ if it satisfies the following relation:

$$[D, X] = \xi X \tag{IV.2}$$

To derive scaling laws for the times, we consider the standard dilation vector field, defined as:

$$D = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \tag{IV.3}$$

Using the inherent symmetries of homogeneous vector fields we can infer the evolution of the execution times.

Theorem 4.3: [AT08b] Consider the control system (II.1) and a feedback law rendering the closed loop system (III.2) homogeneous with degree function $\xi : \mathbb{R}^n \to \mathbb{R}$ with respect to the standard dilation vector field. The execution times $\tau : \mathbb{R}^n \to \mathbb{R}$, implicitly defined by the execution rule $|e| = c|x|$, with $c > 0$, scale according to:

$$\tau(\mathbf{e}^s x) = \mathbf{e}^{-\rho(s)}\tau(x) \quad \forall s \in \mathbb{R} \tag{IV.4}$$

with:

$$\rho(s) = \int_0^s \xi(\mathbf{e}^v x)dv \tag{IV.5}$$

and where $x \in \mathbb{R}^n$ represents any point in the state space. When the degree function ξ is constant, the scaling law (IV.4) simplifies to:

$$\tau(\lambda x) = \lambda^{-\xi}\tau(x), \qquad \lambda = \mathbf{e}^s \tag{IV.6}$$

B. Polynomial systems

We start with the formal definition of a polynomial system.

Definition 4.4: A differential equation:

$$\dot{\mathrm{x}} = f(\mathrm{x}) \tag{IV.7}$$

is said to be polynomial if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial function, *i.e.*, if f satisfies the following relation:

$$f_i(x) = \textstyle\sum_{j=1}^{p} \alpha_{ij} x_1^{r_{1_{ij}}} \ldots x_n^{r_{n_{ij}}} \tag{IV.8}$$

for $p \in \mathbb{N}_{\geq 1}$, $r_{k_{ij}} \in \mathbb{N}_0$, $k = 1, .., n$, $i = 1, \ldots, n$, and $\alpha_{ij} \in \mathbb{R}$.

As explained for example in [Bai80], any polynomial vector field of dimension n can be rendered homogeneous by introducing another state variable $w \in \mathbb{R}$, satisfying $\dot{\mathrm{w}} = 0$. We define l as the highest degree in any of the monomials of f_i for any $i = 1, .., n$. Each monomial of degree $m < l$ is multiplied by w^{l-m}. Thus each component f_i becomes a homogeneous polynomial of degree $l-1$ in the variables $x_1, .., x_n, w$. Hence the state space representation of the extended system is:

$$\begin{aligned} \dot{\mathrm{x}} &= \tilde{f}(\mathrm{x}, \mathrm{w}) \\ \dot{\mathrm{w}} &= 0 \end{aligned} \tag{IV.9}$$

with

$$\tilde{f}_i(x, w) = \sum_{j=1}^{p} \alpha_{ij} x_1^{r_{1_{ij}}} \dots x_n^{r_{n_{ij}}} w^{l-m_{ij}}$$

$$i = 1, \dots, n \quad m_{ij} = \sum_{k=1}^{n} r_{k_{ij}} \quad l = \max_{i,j} m_{ij} \tag{IV.10}$$

and $\mathrm{w}(0) = 1$ so that $\tilde{f}(x, w) = f(x)$. Using this procedure, the system (IV.9) is homogeneous with respect to the standard dilation, and thus we can obtain a scaling law for polynomial systems.

Theorem 4.5: [AT08b] Let $\tau(x)$ be the execution times for a polynomial system $\dot{\mathrm{x}} = f(\mathrm{x})$ implicitly defined by $|e| = c|x|$ for a point $x \in \mathbb{R}^n$, and let $\tilde{\tau}(x, w)$ be the execution times for the system (IV.9) under the same execution rule, for a point $(x, w) \in \mathbb{R}^{n+1}$. Then, τ and $\tilde{\tau}$ are related according to:

$$\tau(x) = \lambda^{l-1} \ \tilde{\tau}(\lambda x, \lambda), \qquad \forall \lambda > 0 \tag{IV.11}$$

with l as defined in (IV.10).

These theorems provide a way to construct a self-trigger condition of the form $\tau = \varphi(x)$. Once the execution time at a given point is known, we can infer the execution times for any state lying on the same ray, according to equations (IV.4) and (IV.11). Hence, in order to compute the execution times for the whole operating region it would be enough to find a lower bound for the times on a sphere and then extend the times along rays. This was the approach followed in [AT08b]. Unfortunately it introduces two sources of conservativeness:

- The existing methods to compute a stabilizing period for a given set (also known in the literature as maximum allowable transfer interval, MATI) are rather conservative ([CTN07], [Tab07]), since in general they are based on the Lipschitz constant of the vector field.
- The stabilizing period represents a lower bound for the execution times of all the points in the sphere. Since times might vary drastically along the sphere (or along any given set), this lower bound will be overly conservative for some points in the sphere. In other words, the evolution of the times across rays is neglected in this procedure.

In the next section we overcome these two drawbacks through the use of isochronous manifolds.

V. Isochronous manifolds

Since polynomial vector fields can be cast as homogeneous vector fields in a higher dimensional space, in this section we only consider homogeneous systems for clarity of exposition. The sphere was selected in [AT08b] as the reference manifold (i.e., the set where a lower bound for the execution times is computed) for simplicity. However, there is no reason to expect that times are constant for the points in the sphere. Instead, one should consider isochronous manifolds which consist of all the points where times remain constant.

It is important to notice that we are interested in isochronous manifolds that allow us to derive global self-trigger conditions for the whole operating region through the scaling laws of section IV. In other words, in order to be of any use, an isochronous manifold should intersect *every* homogeneous ray at least at one point. We prove the existence of this class of isochronous manifolds for homogeneous systems of constant degree in the following proposition.

Proposition 5.1: Consider the control system (II.1) and a feedback law rendering the closed loop (III.2) homogeneous with constant degree function $\xi > 0$. For any given time $t_* \in \mathbb{R}_0^+$ there exists an isochronous manifold Ω_{t_*} of dimension $n-1$ defined by:

$$\Omega_{t_*} = \{x_* \in \mathbb{R}^n : \tau(x_*) = t_*\} \tag{V.1}$$

with τ as defined in (III.6).

Proof: The equation of a ray is:

$$x = \mathbf{e}^s x_0 \qquad s \in \mathbb{R}, \ x_0 \in \mathbb{R}^n \tag{V.2}$$

Since the function ξ is constant, equation (IV.4) simplifies to:

$$\tau(\mathbf{e}^s x) = \mathbf{e}^{-\xi s} \tau(x) \quad \forall s \in \mathbb{R}, \xi \in \mathbb{R} \tag{V.3}$$

According to (V.3), execution times vary from 0 to ∞ as a point moves along a ray, that is, as s varies from $-\infty$ to ∞ in (V.2). Hence, for any time $t_* \in \mathbb{R}_0^+$, there exists a point x_* in each ray such that $\tau(x_*) = t_*$. Moreover, equation (V.3) implies that there does not exist two different points lying on the same ray with the same execution time. The union of all those points x constitute an isochronous manifold for the homogeneous system. Due to space limitations we omit the proof showing that Ω_{t_*} is indeed a manifold. ■

It is easy to see that such manifold can be used instead of the sphere in order to extend the execution times for the whole operating region, as there exists a unique intersection point between each homogeneous ray and the manifold.

Remark 5.2: Note that the proof cannot be extended to the case of state-dependent homogeneous systems for any given t, since times do not need to vary from 0 to ∞ in general. The existence of isochronous manifolds for state-dependent homogeneous systems thus remains an open problem.

VI. Approximating Isochronous manifolds

The isochronous manifold Ω_{t_*} implicitly defined by equation (V.1) represents the set of all states x_* that satisfy the triggering condition (III.5) at time t_*:

$$|\mathrm{e}(t_*, x_*)| = \sigma \frac{a}{b} |\mathrm{x}(t_*, x_*)| \tag{VI.1}$$

The explicit computation of the manifold would involve the knowledge of the flow, which in general unknown. Herein we develop a technique to derive an approximation of the isochronous manifold. To avoid singularities in the computations, we rewrite (III.5) as:

$$\phi(x) := e^T e - \left(\sigma \frac{a}{b}\right)^2 x^T x = 0 \tag{VI.2}$$

Since we are only interested in the evolution of the composition $\phi \circ \mathrm{x}(t)$, it would be sufficient to construct a differential equation for $\phi \circ \mathrm{x}(t)$. In [Tab07] a first order nonlinear differential equation of the form $\frac{d}{dt}(\phi \circ \mathrm{x}(t)) = g(\phi \circ \mathrm{x}(t))$ was derived, whose solution just depended on the norm of the state x. Instead, we derive an nth order differential equation, whose solution collects more information from the state. Moreover, in order to obtain a closed-form expression for its solution, we focus on linear differential equations describing the evolution of $\phi \circ \mathrm{x}(t)$. Clearly, $\phi \circ \mathrm{x}(t)$ will be described by a linear differential equation only under special circumstances, see *e.g.* [LM86]. However, the evolution of $\phi \circ \mathrm{x}(t)$ can be *bounded* by a linear differential equation. This amounts to constructing a set of coefficients $\delta_i \in \mathbb{R}$ satisfying:

$$(\mathcal{L}_X^n \phi)(x) \leq \sum_{i=0}^{n-1} \delta_i (\mathcal{L}_X^i \phi)(x) \tag{VI.3}$$

for any state x in a region of interest $\Gamma \subseteq \mathbb{R}^n$ as stated in the next result.

Lemma 6.1: Consider a vector field $X : M \to TM$ and a map $h : M \to \mathbb{R}$. If there exists a sequence of coefficients $\delta_0, \delta_1, \ldots \delta_{n-1} \in \mathbb{R}$ satisfying:

$$(\mathcal{L}_X^n h)(x) \leq \sum_{i=0}^{n-1} \delta_i (\mathcal{L}_X^i h)(x) \qquad x \in M \tag{VI.4}$$

then the following implication holds for all $x_0 \in M$ and for all $t \in \mathbb{R}_0^+$ for which the solution ψ of X is defined:

$$h \circ \psi(t, x_0) \leq \mathrm{y}_1(t, y_0) \tag{VI.5}$$

where y_1 is the first component of the solution of the linear differential equation:

$$\dot{\mathrm{y}}_i = \mathrm{y}_{i+1} \qquad i = 1, \ldots, n-1 \tag{VI.6}$$

$$\dot{\mathrm{y}}_n = \sum_{i=0}^{n-1} \delta_i \mathrm{y}_i \tag{VI.7}$$

with initial condition:

$$y_{0i} = (\mathcal{L}_X^{i-1} \phi)(x_0) \qquad i = 1, \ldots, n.$$

Proof: By the Comparison Lemma [Kha02], it is easy to see that $(\mathcal{L}_X^{n-1} h) \circ \psi(t, x_0) \leq \mathrm{y}_n(t, y_0)$. Applying the Lemma again successively, we obtain an upper bound for the evolution of h:

$$h \circ \psi(t, x_0) \leq \mathrm{y}_1(t, y_0) \qquad t \geq 0 \tag{VI.8}$$

■

Using this lemma we can see that the minimum time satisfying $\mathrm{y}_1(t, y_0) = 0$ lower bounds the minimum time satisfying $\phi \circ \mathrm{x}(t, x_0) = 0$, since $\phi \circ \mathrm{x}(0, x_0) \leq 0$. Thus, the execution times defined by $\mathrm{y}_1(t, x) = 0$ represent a lower bound on the execution times τ defined by the triggering condition (VI.2). We can interpret this lemma as a way to construct a linear dynamical system $\dot{\mathrm{y}} = A\mathrm{y}$ that describes the evolution of ϕ, with A defined as:

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ \delta_0 & \delta_1 & \delta_2 & \ldots & \delta_{n-2} & \delta_{n-1} \end{bmatrix} \tag{VI.9}$$

Using this notion, we rewrite the triggering condition $\mathrm{y}_1(t, x) = 0$ as:

$$\mathrm{y}_1(t, x) = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} e^{At} \begin{bmatrix} \phi(x) \\ (\mathcal{L}_X \phi)(x) \\ \vdots \\ (\mathcal{L}_X^{n-1} \phi)(x) \end{bmatrix} = 0 \tag{VI.10}$$

It is clear that by fixing $t = t_*$ in equation (VI.10) we obtain an equation describing the set of states whose execution times are lower bounded by t_*. Hence we can define an approximation for the isochronous manifolds Ω_{t_*} as:

$$\tilde{\Omega}_{t_*} = \{x_* \in \mathbb{R}^n : \mathrm{y}_1(t_*, x_*) = 0\} \tag{VI.11}$$

In order to apply the scaling laws (IV.4) and (IV.11) with $\tilde{\Omega}$, we first find the intersection between the homogeneous rays and $\tilde{\Omega}$. Since we are searching for a self-trigger condition to be applied online, it is desirable to have a closed form expression for those intersecting points. Towards this objective, we state a simple lemma describing a useful property of the Lie derivative of ϕ along homogeneous vector fields.

Lemma 6.2: Consider a map $\phi : M \to \mathbb{R}$ homogeneous of degree d and a vector field $X : M \to TM$ homogeneous of degree ξ. Then, the k-th Lie derivative of ϕ along X is homogeneous of degree $d + k\xi$:

$$(\mathcal{L}_X^k \phi)(\lambda x) = \lambda^{d+k\xi} (\mathcal{L}_X^k \phi)(x), \qquad x \in M, \lambda > 0 \tag{VI.12}$$

The result can be easily proven by induction. Notice that ϕ as defined in (VI.2) is homogeneous of degree 1. Let x_{t_j} be the last measurement of the state. To make use of the scaling law (IV.6), we identify λx with x_{t_j} and x with a point in $\tilde{\Omega}$. To compute the intersection point between a ray passing through x_{t_j} and $\tilde{\Omega}$ we substitute the equation of the ray $x_{t_j} = \lambda x_*$ in the triggering condition (VI.10):

$$\begin{aligned} \mathrm{y}_1(t_*, x_*) &= \mathrm{y}_1(t_*, \frac{1}{\lambda} x_{t_j}) \\ &= \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} e^{At_*} \begin{bmatrix} \phi(\frac{1}{\lambda} x_{t_j}) \\ (\mathcal{L}_X \phi)(x)|_{x=\frac{1}{\lambda} x_{t_j}} \\ \vdots \\ (\mathcal{L}_X^{n-1} \phi)(x)|_{x=\frac{1}{\lambda} x_{t_j}} \end{bmatrix} \\ &= \sum_{i=0}^{n-1} \beta_i(t_*, x_{t_j}) \lambda^{-(2+i\xi)} = 0 \end{aligned} \tag{VI.13}$$

with:

$$\beta_i(t_*, x_{t_j}) = {e^{At_*}}_{1(i+1)} \;\; (\mathcal{L}_X^i \phi)(x_{t_j}) \qquad i = 0, \ldots, n-1 \tag{VI.14}$$

Since we are interested in the unique positive real solution, we can multiply equation (VI.13) by $\lambda^{2+(n-1)\xi}$ and define the variable $z = \lambda^{\xi}$ to obtain the equivalent equation:

$$\sum_{i=0}^{n-1} \beta_i z^{n-1-i} = 0 \tag{VI.15}$$

This equation reflects the tradeoff between complexity and accuracy of the execution times: the bigger n is, the better approximation will be attained. The value $n = 3$ seems to be a sensible choice since it allows us to find an easy analytical solution for (VI.15). This solution can be substituted into equation (IV.6) to obtain a self-trigger condition, as summarised in the following theorem.

Theorem 6.3: Consider the control system (II.1) and a feedback law rendering the closed loop system (III.2) homogeneous with constant degree function $\xi > 0$. If there exists a sequence of coefficients $\delta_0, \delta_1, \ldots \delta_k \in \mathbb{R}$ satisfying (VI.3), then for any $t_* \in \mathbb{R}_0^+$ the submanifold $\tilde{\Omega}_{t_*}$, defined by (VI.11), upper bounds the isochronous manifold Ω_{t_*} in the following sense:

$$x \in \tilde{\Omega}_{t_*} \quad \Longrightarrow \quad \tau(x) \geq t_*$$

Moreover, when $k = 3$ the function $\tau^{\downarrow} : M \to \mathbb{R}$ defined by:

$$\tau^{\downarrow}(x) = \frac{2\beta_0}{-\beta_1 + \operatorname{sign}(\beta_0)\sqrt{\beta_1^2 - 4\beta_2\beta_0}}\; t_* \qquad \text{(VI.16)}$$

provides a lower bound for the execution times $\tau(x)$ with $\beta_i(t_*, x)$ given by (VI.14).

Remark 6.4: For polynomial systems, a lower bound for the execution times $\tau(x)$ is also given by (VI.16), where now the coefficients β_i are computed with the homogenised system as defined in (IV.9). For state-dependent homogeneous system, the procedure can still be applied, but now there is no guarantee that there always exist a intersecting point between any homogeneous ray and the isochronous manifold. Hence, the resulting self-trigger condition might not be valid for the whole operating region.

VII. Examples

We compare the results herein developed with our previous work in [AT08a] and [AT08b]. For reasons of space we only cover in detail the example in [AT08a], and we just summarise the results for the rigid body example appearing in [AT08b]. The equations of the example in [AT08a] are:

$$\begin{aligned} \dot{x_1} &= -x_1^3 + x_1 x_2^2 \\ \dot{x_2} &= x_1 x_2^2 + u - x_1^2 x_2 \end{aligned} \qquad \text{(VII.1)}$$

with $u = -x_2^3 - x_1 x_2^2$. The operating region is a ball of radius 1 around the origin. Using [PPSP04] we find the coefficients δ_i defining the approximation of the isochronous manifold:

$$\delta_0 = 105.970 \quad \delta_1 = 0.021 \quad \delta_2 = 1.033$$

Figure 1 depicts $\tilde{\Omega}_{t_*}$ for $t_* = 1$ms, computed according to equation (VI.11), and the isochronous manifold Ω_{t_*} computed via numerical simulations. As proven in Theorem 6.3, the exact isochronous manifold encloses $\tilde{\Omega}$, since times enlarge as we approach the origin. Hence, two conclusions can be drawn from this figure. First, we notice that $\tilde{\Omega}$ nearly coincides with Ω, while there is a considerable gap between Ω and the sphere, showing that the lower bound developed in [AT08b] was not tight. Moreover, the shape of the isochronous manifold is clearly different from the sphere, implying that even if a tight sphere is computed, there exist many points where the execution times will be significantly conservative.

Since we are not aware of any other work addressing self-trigger strategies for nonlinear systems, in the simulations we compare the execution times defined by Theorem 6.3 against

σ	periodic	self-triggered [AT08b]	self-triggered Thm. (6.3)	event-triggered
0.1	0.39	0.39	1.50	1.55
0.2	0.79	0.79	3.00	3.06
0.3	1.18	1.18	4.50	4.58

TABLE I

Average time for the [AT08a] example (in ms.)

the periodic strategy, the event-triggered times generated by condition (III.5) and the self-trigger technique in [AT08a]. The system exhibits a similar behaviour under the 4 different implementations for all tested initial conditions (not shown here due to the lack of space).

As our previous work in [AT08a] describes the exact evolution of the times along rays, we focus our comparison on the evolution of times across rays, which is the main topic addressed in this paper. For that purpose, we consider 20 initial conditions equally spaced along a sphere. The self-triggered condition developed in [AT08a] defines the execution times just as a function of the norm of the state, hence it will render the same execution times for all points lying on the sphere. On the other hand, Theorem 6.3 takes into account all the information contained in the state. Figure 2 shows the execution times under the 4 strategies as a function of the position in the boundary of the ball (that is, $x = (\cos(\theta), \sin(\theta))$, for $\theta \in [0, 2\pi]$), for $\sigma = 0.3$. Table I represents the average execution time for the 20 points along the boundary for different values of σ (*i.e.*, for different degrees of performance). We can observe both in Table I and Figure 2 how the new proposed technique nearly matches the event-trigger times and improves significantly the times generated by the former self-trigger strategy, which generates the same times as the periodic strategy.

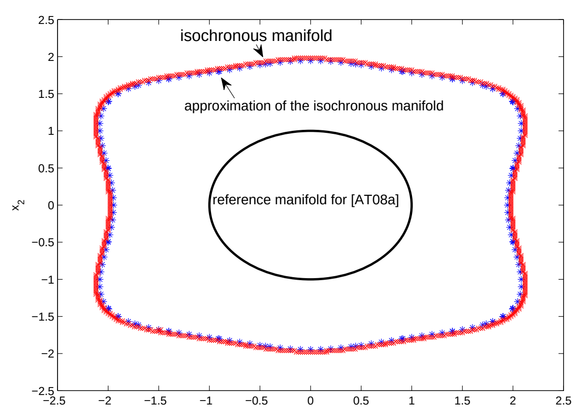

Fig. 1. Reference manifold for [AT08a], isochronous manifold and its approximation according to equation (VI.11).

Finally, Figure 3 shows the evolution of the execution times along trajectories for the 4 different implementations for a particular initial condition ($x_0 = (0.4, 0.7)$) and for a simulation of 5s. Again, we can observe that the new self-trigger technique nearly tracks the evolution of the event-triggered times.

We briefly now consider the rigid body example. The state space representation of such system with two inputs can be

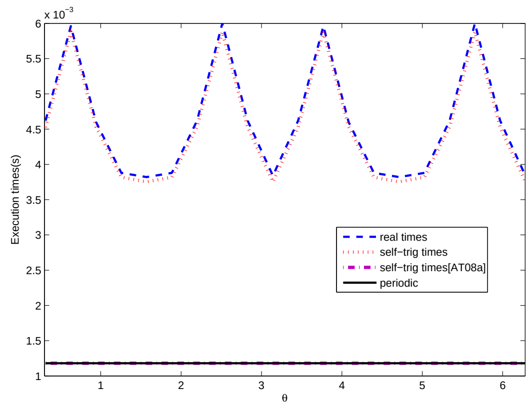

Fig. 2. Execution times along a sphere of unitary radius.

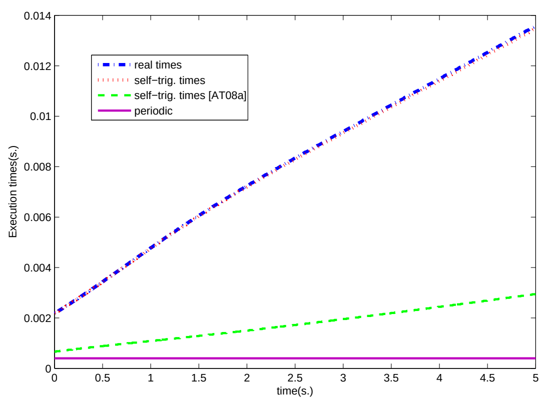

Fig. 3. Evolution of the execution times along trajectories.

simplified to the form:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 x_2 \end{aligned} \tag{VII.2}$$

A nonlinear feedback law is designed in [BI89] to render the system globally asymptotically stable:

$$\begin{aligned} u_1 &= -x_1x_2 - 2x_2x_3 - x_1 - x_3 \\ u_2 &= 2x_1x_2x_3 + 3x_3^2 - x_2 \end{aligned} \tag{VII.3}$$

The closed loop system is polynomial with $l = 3$. Following the same steps as before, we evaluate the proposed self-trigger technique for 20 initial conditions lying on the boundary of a sphere of radius 15. Table II shows the average execution times. Again, the new self-trigger condition nearly matches the execution times rendered by the event-triggered technique, outperforming the previous self-trigger condition by a factor of 20.

Due to lack of space, we did not address in this paper practical issues such as robustness. We refer the interested reader to [MJT09] for a discussion of the topic.

VIII. Discussion

While in this paper we only covered the case of homogeneous and polynomial systems, Lemma 6.1 is applicable to any nonlinear system. The lemma was used in this paper to find an approximation for the isochronous manifold: given

σ	self-triggered [AT08b]	self-triggered Thm. (6.3)	event-triggered
0.1	0.17	3.24	3.36
0.2	0.30	6.50	6.61
0.3	0.40	9.86	9.88

TABLE II
Average time for the rigid body example (in ms.)

a time t_*, we were interested in the set of states x_* that satisfy equation (VI.10). Alternatively, it can be used as well to solve the inverse problem: for a given state x, find the execution time t satisfying (VI.10). This approach provides a way to construct a self-trigger condition for general nonlinear systems. However, equation (VI.10) is transcendental in t and numerical solutions are computationally too expensive to be applied online. Nonetheless, it is possible to apply the procedure in [MJAT09] to (VI.10), *mutatis mutandis*, in order to obtain online a sequence of execution times guaranteeing (III.4). This generalization is currently being studied by the authors.

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