Discrete Synthesis

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CyPhyLab UCLA

Systems

Definitions

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In this workshop we will use the notion of *system* to capture both finite-state as well as infinite-state models of dynamic phenomena.

Definition (System)

A *system* S is a sextuple $(X, X_0, U, \longrightarrow, Y, H)$ consisting of:

- a set of *states* X;
- a set of *initial states* $X_0 \subseteq X$;
- a set of *inputs* U;
- a *transition relation* $\longrightarrow \subseteq X \times U \times X$;
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A system is called *finite-state* if X is a finite set. A system that is not finite-state is called *infinite-state*.

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The evolution of a system is captured by the transition relation. A transition $(x, u, x') \in \longrightarrow$ is denoted by $x \xrightarrow{u} x'$. We can also regard the transition relation as a set-valued map $\delta : X \times U \to 2^X$.

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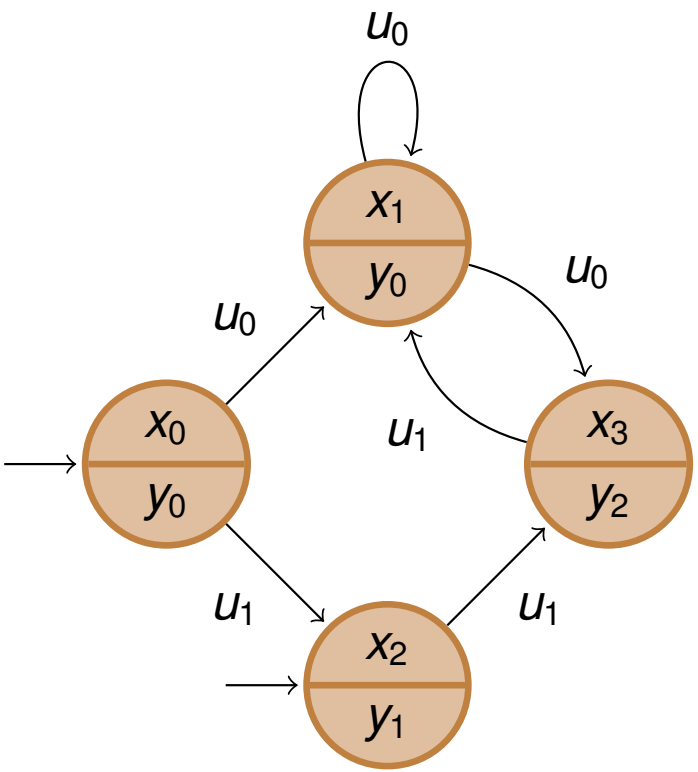

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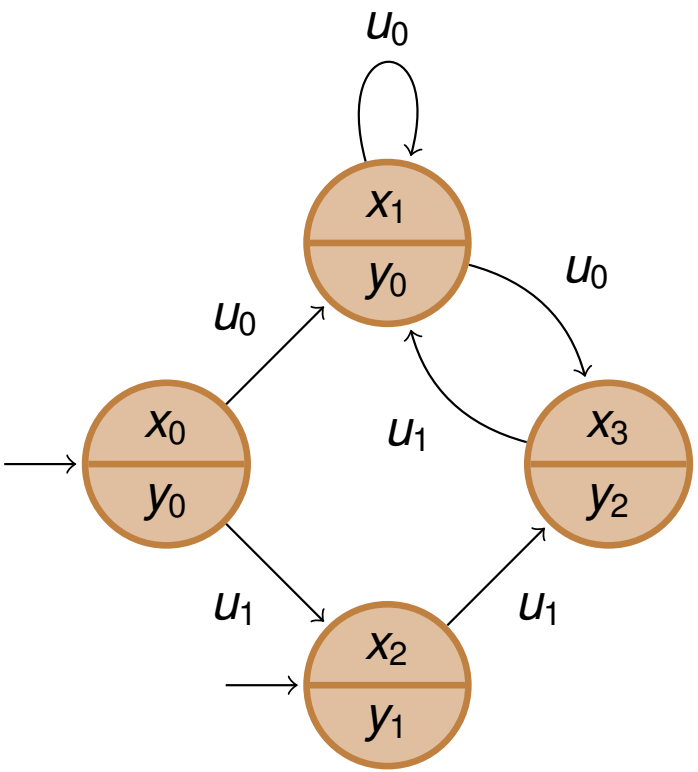

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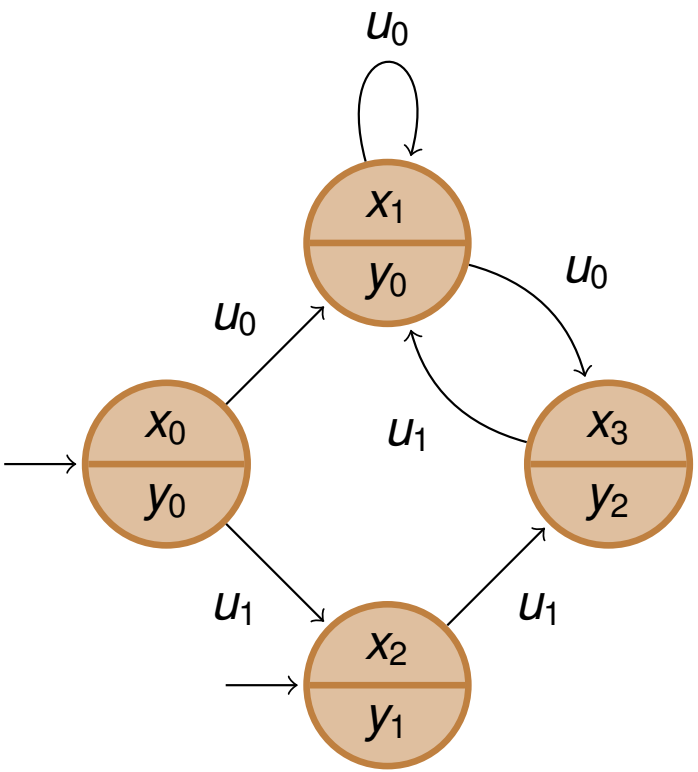

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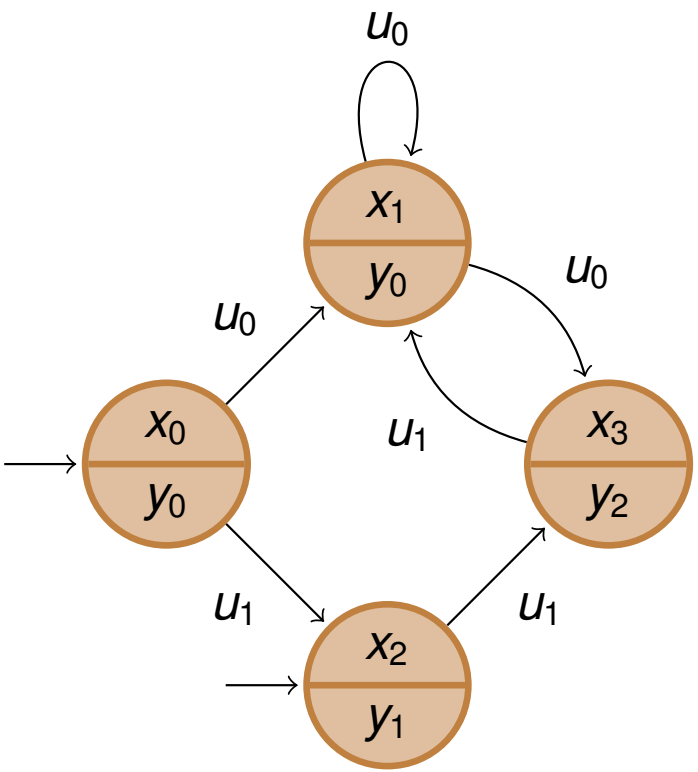

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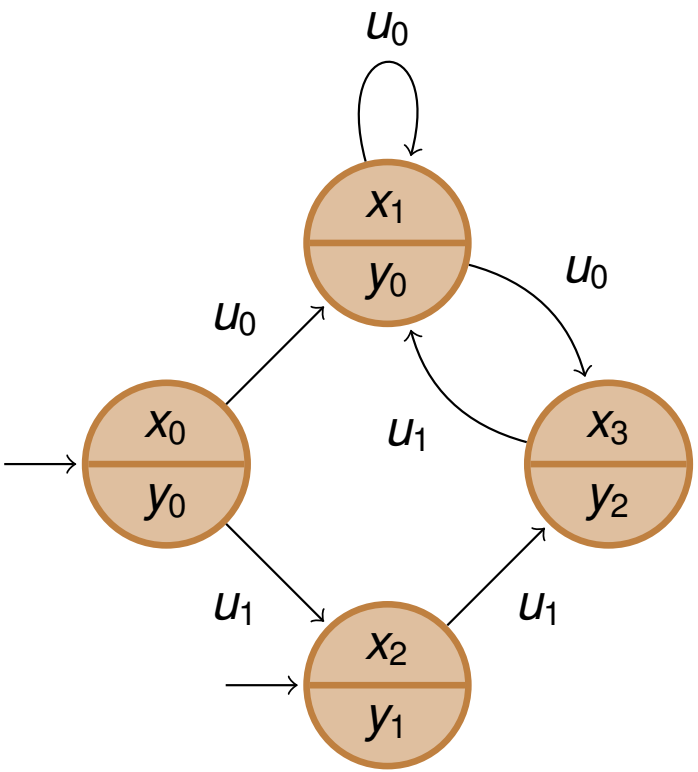

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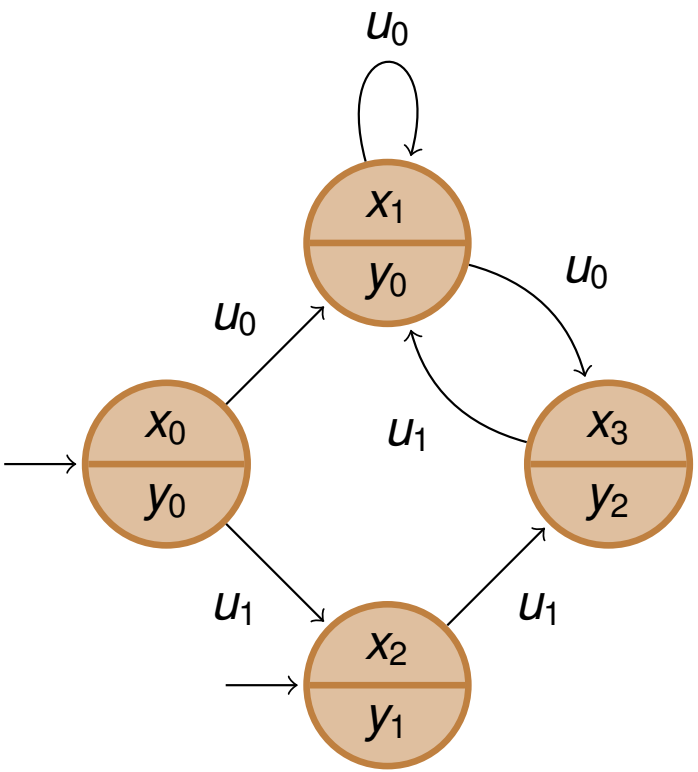

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Y &= \{y_0, y_1, y_2\} \\
H(x_0) &= y_0, \quad H(x_1) = y_0, \quad H(x_2) = y_1 \\
H(x_3) &= y_2.
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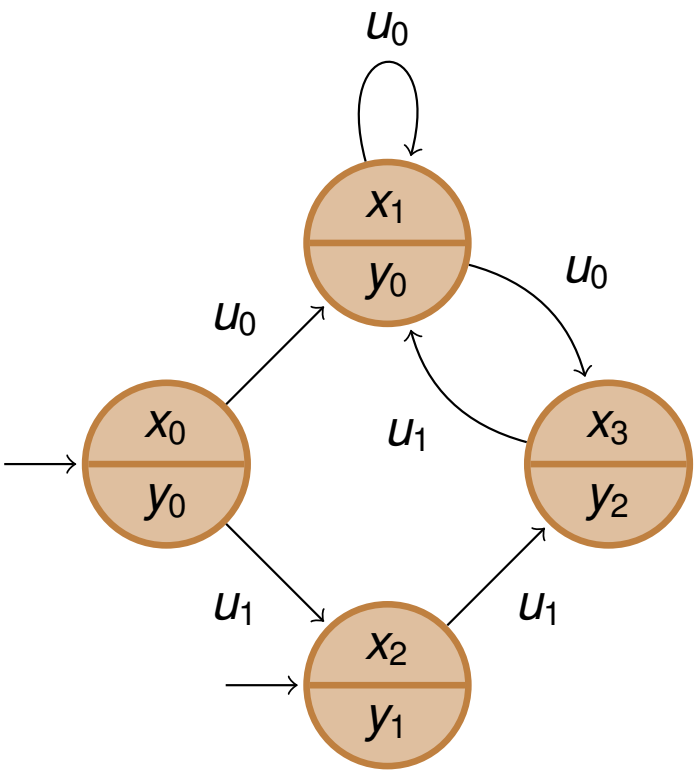

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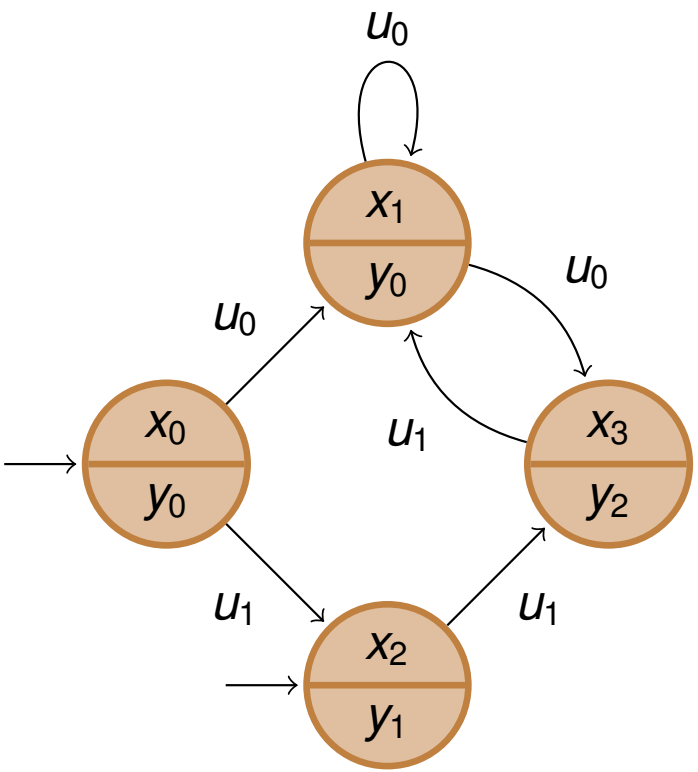

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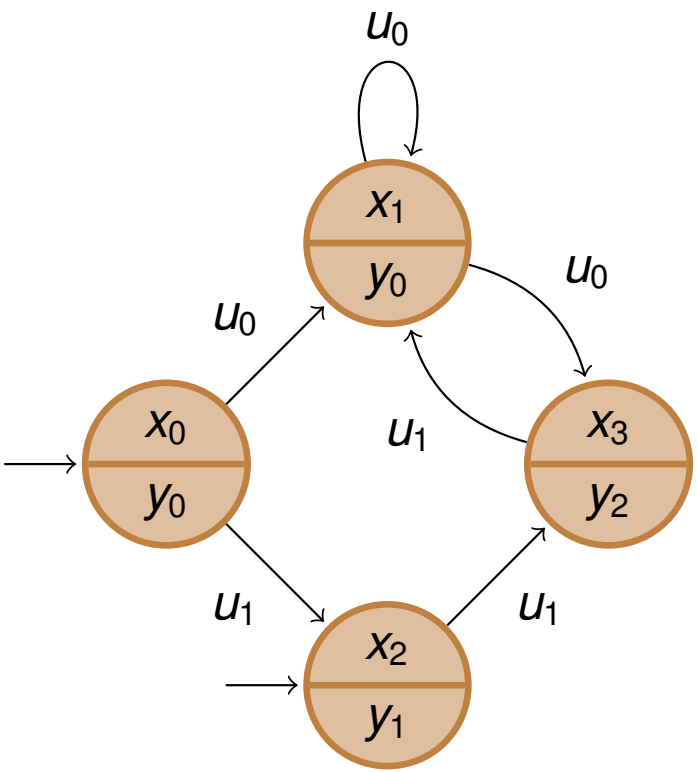

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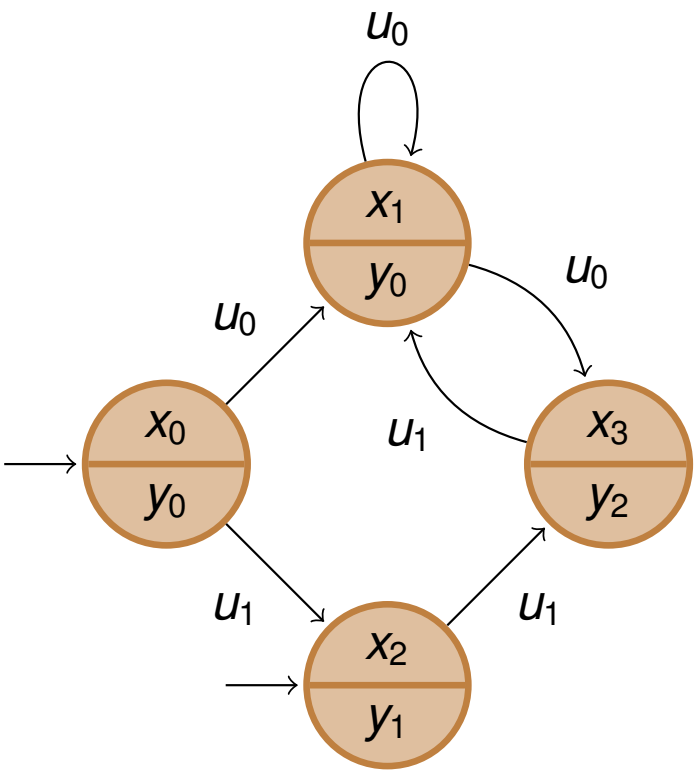
We denote the set of u-successors of a state x by $\mathrm{Post}_u(x)$.



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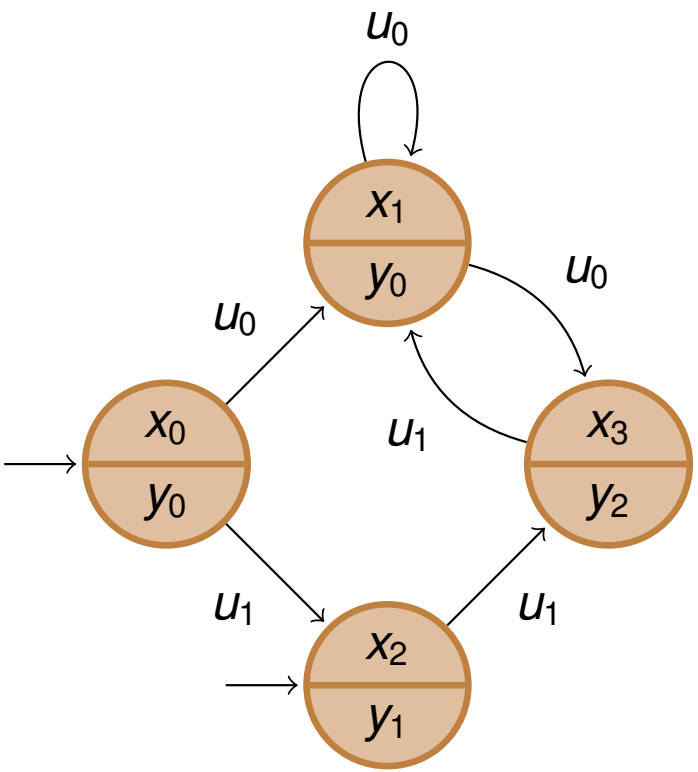
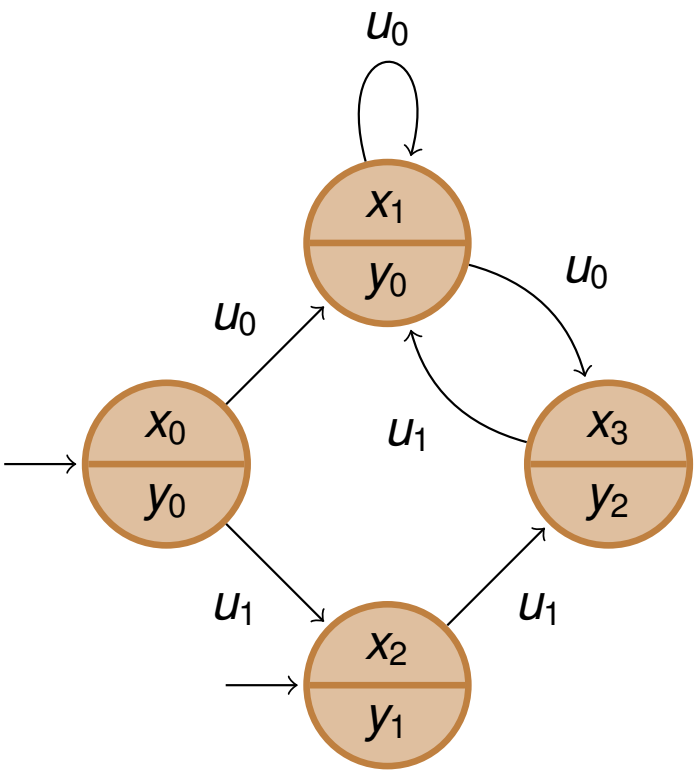
We denote the set of u-successors of a state x by $\mathrm{Post}_u(x)$.

Since $\mathrm{Post}_u(x)$ may be empty, we denote by $U(x)$ the set of inputs $u \in U$ for which $\mathrm{Post}_u(x)$ is nonempty.



This system is nondeterministic since there are two distinct u_0-successors of x_1, i.e. $\mathrm{Post}_{u_0}(x_1) = \{x_1, x_3\}$.



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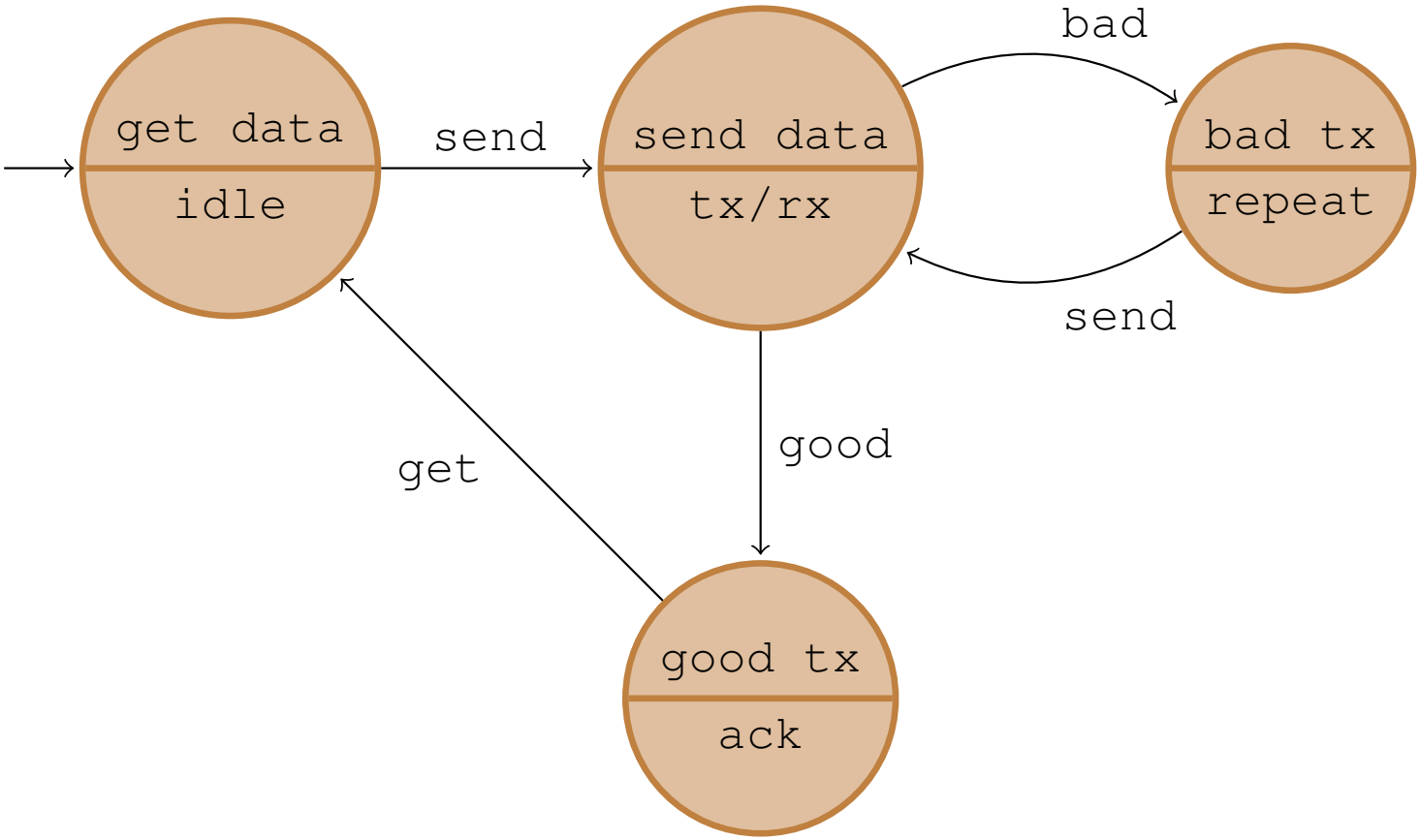
This system is non-blocking since every state has at least one successor, i.e. $U(x) \neq \varnothing$ for every $x \in X$.

A simple communication protocol (transmission side).



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\begin{aligned}
&\texttt{x} := 0\\
&\texttt{n} := 0\\
&\texttt{While}(\textbf{true})\\
&\quad \{\\
&\quad \texttt{y} := \texttt{read}(\texttt{input})\\
&\quad \texttt{x} := \texttt{x}\frac{\texttt{n}}{\texttt{n}+1} + \texttt{y}\frac{1}{\texttt{n}+1}\\
&\quad \texttt{n} := \texttt{n}+1\\
&\quad \}
\end{aligned}
$$

The variable *y* contains the latest received number and the variable *x* contains the average of the numbers that have been received so far.

Assume now that we are interested in knowing if x is smaller, equal, or greater than 1 when y is restricted to assume values in the set $\{1, 2\}$.

$$\texttt{x} := 0$$

$$\texttt{n} := 0$$

$$\texttt{While}(\textbf{true})$$

$$\{$$

$$\texttt{y} := \texttt{read}(\texttt{input})$$

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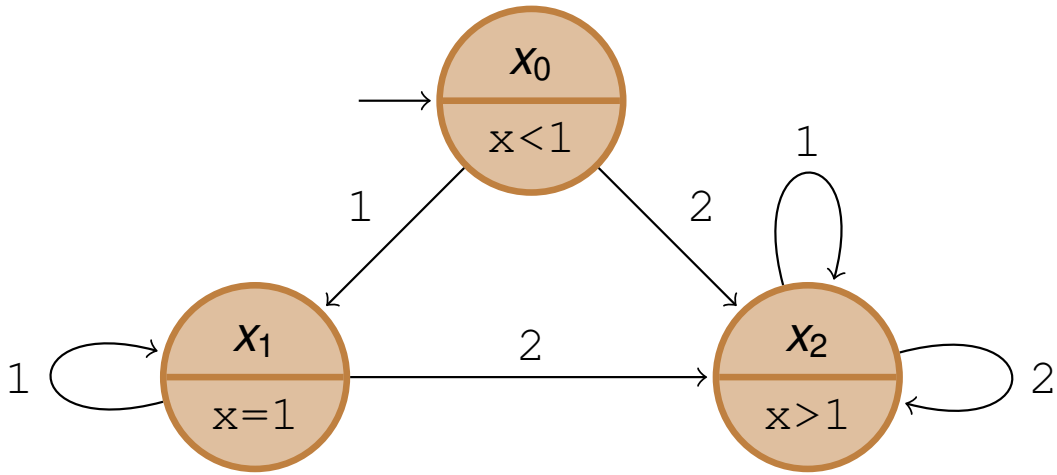
Assume now that we are interested in knowing if x is smaller, equal, or greater than 1 when y is restricted to assume values in the set $\{1, 2\}$.

```
x := 0
n := 0
While(true)
    {
    y := read(input)
    x := x n/(n+1) + y 1/(n+1)
    n := n + 1
    }
```



Dynamical systems evolving in continuous-time can also be modeled as systems. Consider Euler's equations for the rotational dynamics of a rigid body around its center of mass:

$$\frac{d}{dt}\xi_1 = \frac{I_2 - I_3}{I_1}\xi_2\xi_3 \quad (1)$$

$$\frac{d}{dt}\xi_2 = \frac{I_3 - I_1}{I_2}\xi_3\xi_1 \quad (2)$$

$$\frac{d}{dt}\xi_3 = \frac{I_1 - I_2}{I_3}\xi_1\xi_2 \quad (3)$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ is the body angular velocity and $I_1, I_2, I_3 \in \mathbb{R}$ are the principal moments of inertia.

The rigid body can be described by the system $S = (X, X_0, U, \longrightarrow, Y, H)$ consisting of:

- $X = \mathbb{R}^3$;
- $X_0 = X$;
- $U = \mathbb{R}_0^+$;
- $x \xrightarrow{\tau} x'$ if there exists a solution ξ to equations (1) through (3) satisfying $\xi(0) = x$ and $\xi(\tau) = x'$;
- $Y = X$;
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Several other classes of systems can be conveniently described by the proposed notion of system. Examples include timed-automata, switched systems, hybrid systems, and many others.

Given a system $S = (X, X_0, U, \longrightarrow, Y, H)$ and a state $x \in X$, a *finite internal behavior* generated from x is a finite sequence of transitions:

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} \ldots \xrightarrow{u_{n-2}} x_{n-1} \xrightarrow{u_{n-1}} x_n$$

such that $x_0 = x$ and $x_i \xrightarrow{u_i} x_{i+1}$ for all $0 \leq i < n$.

An internal behavior generated from x is *initialized* if $x \in X_0$.

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An internal behavior generated from x is *initialized* if $x \in X_0$.

In some cases, a finite internal behavior can be extended to an infinite internal behavior. An *infinite internal behavior* generated from x is an infinite sequence:

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} x_3 \xrightarrow{u_3} \dots$$

that satisfies $x_0 = x$ and $x_i \xrightarrow{u_i} x_{i+1}$ for all $i \in \mathbb{N}_0$. An infinite internal behavior generated from x is called *initialized* if $x \in X_0$.

In nonblocking systems, every finite internal behavior can be extended to an infinite internal behavior.

Through the output map, every internal behavior:

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-2}} x_{n-1} \xrightarrow{u_{n-1}} x_n$$

defines an external behavior:

$$y_0 \longrightarrow y_1 \longrightarrow y_2 \longrightarrow \cdots \longrightarrow y_{n-1} \longrightarrow y_n$$

with $H(x_i) = y_i \in Y$ for all $0 \leq i \leq n$.

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The set of external behaviors that are defined by internal behaviors generated from state x is denoted by $\mathcal{B}_x(S)$ and is called the *external behavior* from state x.

Definition (Finite External Behavior)

The *finite external behavior* generated by a system S, denoted by $\mathcal{B}(S)$, is defined by:

$$\mathcal{B}(S) = \bigcup_{x \in X_0} \mathcal{B}_x(S).$$

An infinite internal behavior from x:

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The set of all infinite external behaviors that are generated from x is denoted by $\mathcal{B}^\omega_x(S)$ and called the *infinite external behavior* from state x.

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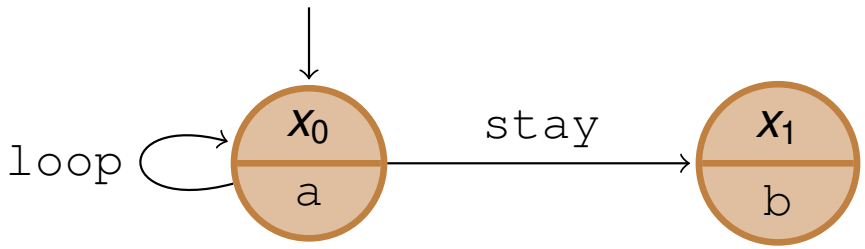
The *infinite external behavior* generated by a system S, denoted by $\mathcal{B}^\omega(S)$, is defined by:

$$\mathcal{B}^\omega(S) = \bigcup_{x \in X_0} \mathcal{B}^\omega_x(S).$$

If a system S is non-blocking, then $\mathcal{B}^\omega(S)$ is nonempty. However, $\mathcal{B}^\omega(S)$ may be nonempty even if S is a blocking system.



The infinite external behavior $\texttt{aaaaa}\ldots$ belongs to $\mathcal{B}^\omega(S)$ although S is blocking since the state x_1 has no successors.

Discrete Synthesis

Safety games

The first class of synthesis problems that we consider are safety games.

We are given a system S and a set of safe outputs $W \subseteq Y$. The objective is to synthesize a controller C such that S_C is nonblocking and $\mathcal{B}^\omega(S_C) \subseteq W^\omega$.

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$$x \xrightarrow[C]{u} x' \quad \text{if} \quad x \xrightarrow{u} x' \wedge u \in C(x).$$

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In other words, we restrict $U(x)$ to $U(x) \cap C(x)$.

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Definition (Safety game)

Let S be a system with $Y = X$ and $H = 1_X$, and let $W \subseteq X$ be a set of safe states. The *safety game* for system S and specification set W asks for the existence of a controller C such that:

1. S_C is nonblocking;
2. $\varnothing \neq \mathcal{B}^\omega(S_C) \subseteq W^\omega$.

A safety game is said to be solvable when S_C exists.

The requirement $Y = X$ and $H = 1_X$ is made without loss of generality since the general case where $Y \neq X$ can be reduced to this one.

Safety games can be solved by constructing a suitable operator:

$$F_W : 2^X \to 2^X$$

for any specification set $W \subseteq X$:

$$F_W(Z) = \{x \in Z \mid x \in W \text{ and } \exists u \in U(x) \quad \varnothing \neq \text{Post}_u(x) \subseteq Z\}.$$

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A fixed-point for F_W is a set $Z \subseteq X$ satisfying $F_W(Z) = Z$. Such fixed-point defines a set Z of states from which it is possible to control system S so as to remain in W.

Proposition

Let S be a system with $Y = X$ and $H = 1_X$, and let $W \subseteq X$ be a set of safe states. The safety game for system S and specification set W is solvable iff the maximal fixed-point Z of F_W satisfies $X_0 \subseteq Z$.

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Since $X_0 \subseteq Z$ we can start at a state in X_0. Then, by using any input given by C we move to a new state still in Z. Hence, we can stay in $Z \subseteq W$ forever.

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Will this controller work?

Since $X_0 \subseteq Z$ we can start at a state in X_0. Then, by using any input given by C we move to a new state still in Z. Hence, we can stay in $Z \subseteq W$ forever.

If we have the possibility to set the initial state, then we can relax $X_0 \subseteq Z$ to $Z \cap X_0 \neq \varnothing$ so that there exists at least one initial state belonging to Z.

A complete characterization of the solutions to safety games can now be obtained by noting that the maximal fixed-point of F_W can be obtained by iteration.

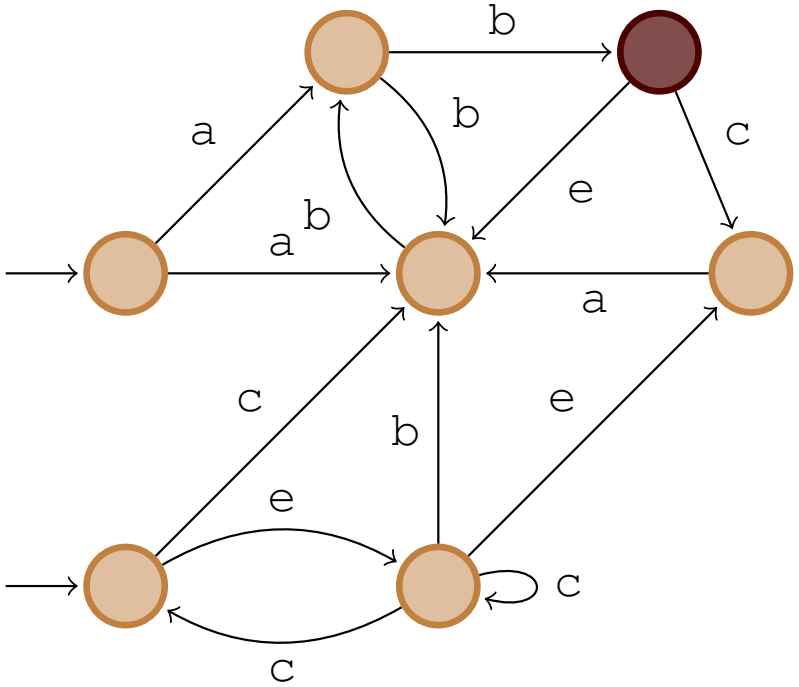
Theorem

Let S be a system with $Y = X$ and $H = 1_X$, and let $W \subseteq X$ be a set of safe states. The safety game for system S and specification set W is solvable iff the maximal fixed-point Z of the operator F_W satisfies $X_0 \subseteq Z$. Moreover, Z can be obtained as:

$$Z = \lim_{i \to \infty} F_W^i(X).$$

When $X_0 \subseteq Z$, a solution to the safety game is given by the controller (4).

Example: synthesize a controller forcing the closed-loop system not to enter the darker state.



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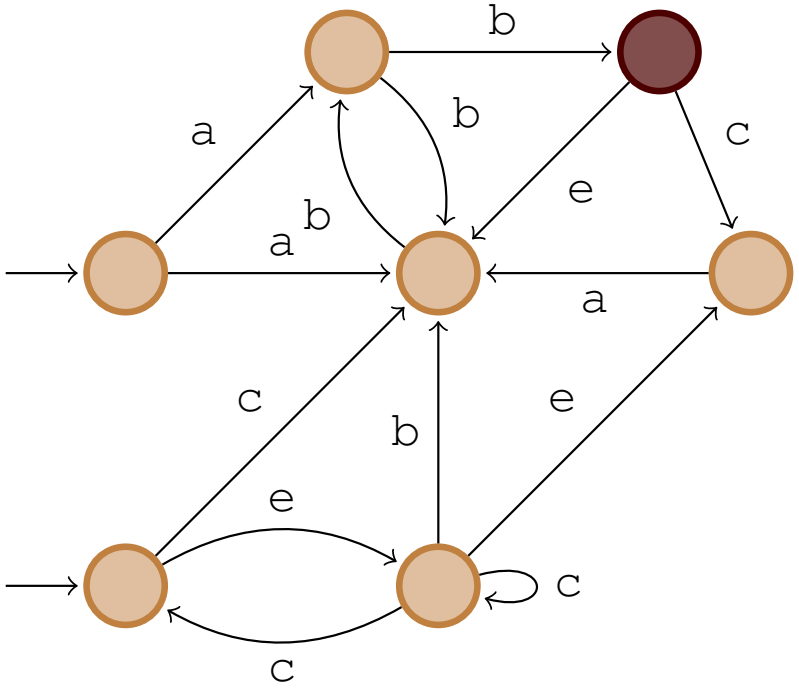

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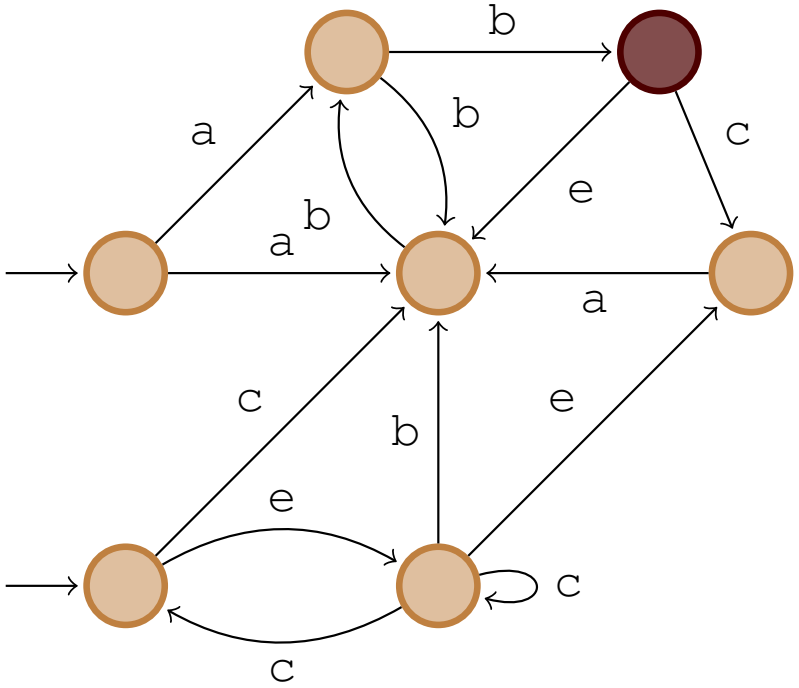

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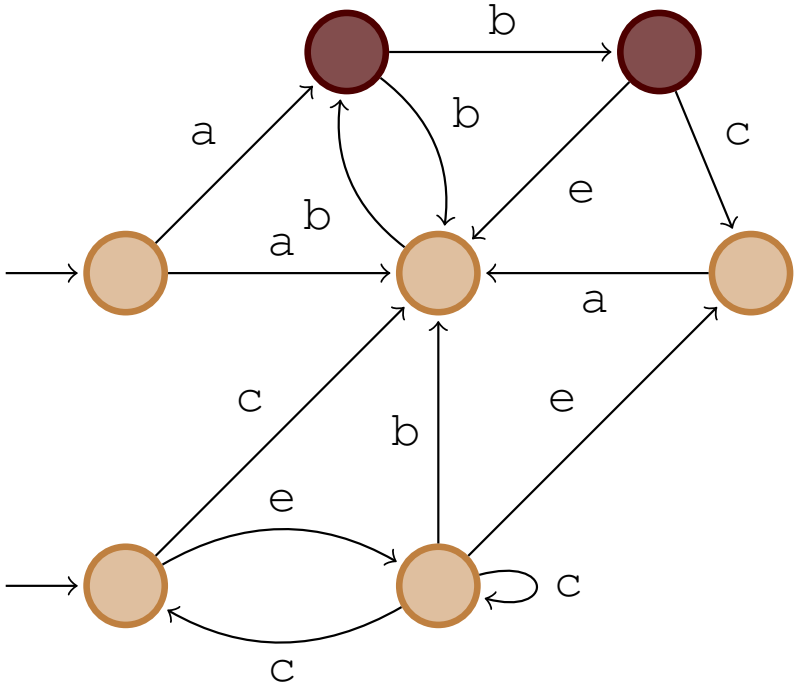

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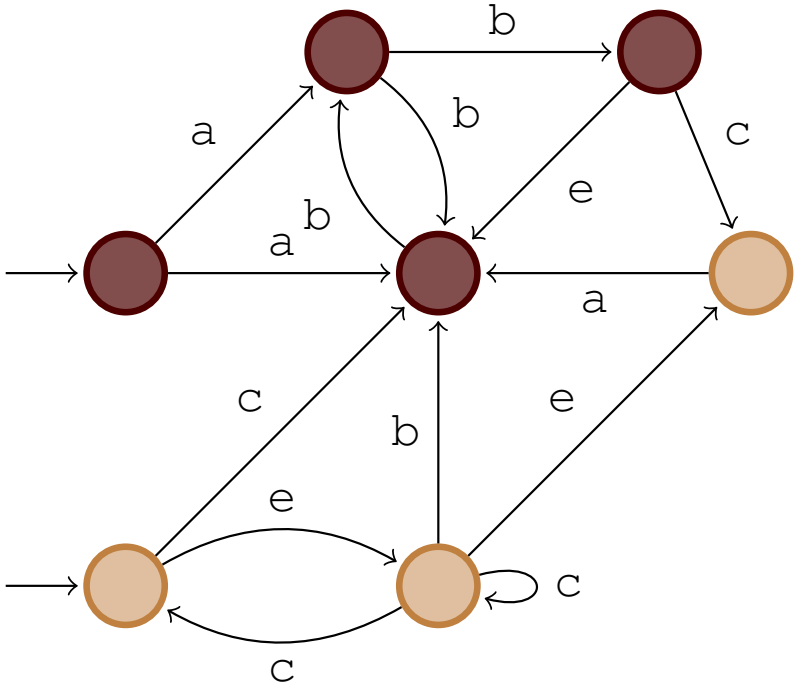

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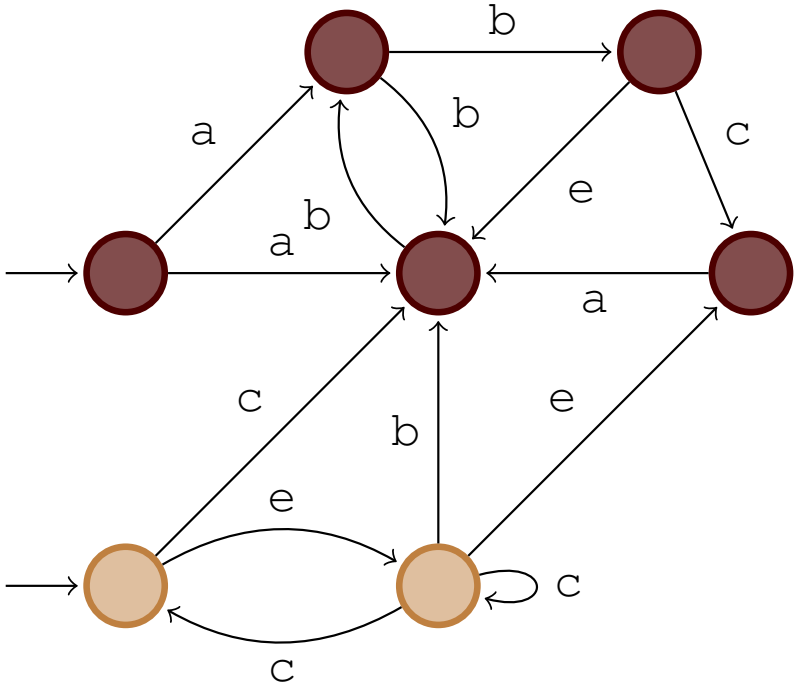

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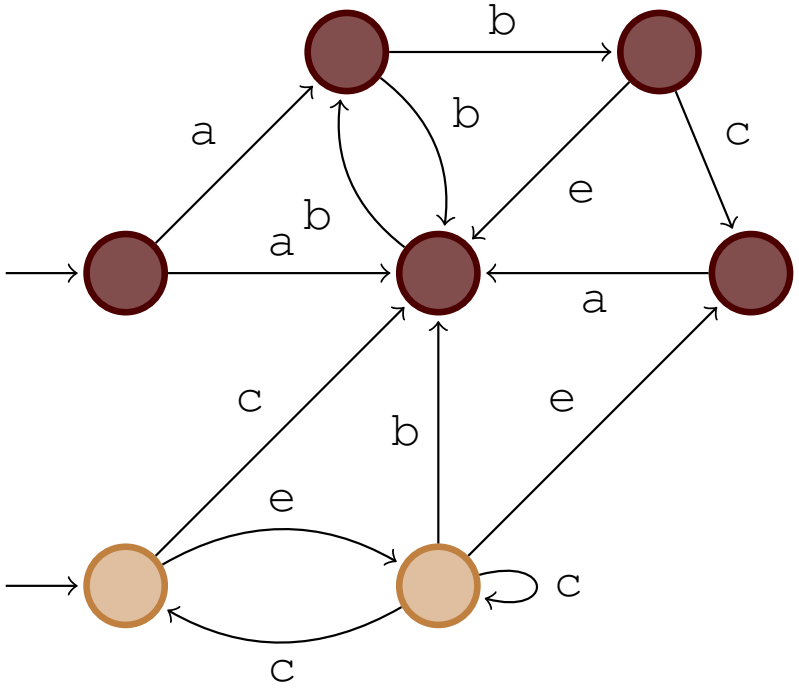

$$F_W^5(X)$$

If we use the maximal fixed point of F_W then we have the following maximality property for the corresponding controller:

Proposition

Let S be a system with $Y = X$ and $H = 1_X$, and let $W \subseteq X$ be a set of safe states. For any controller D solving the safety game for system S and specification set W and for any $x \in Z$ we have:

$$D(x) \subseteq C(x)$$

where C is the controller (4) defined by the maximal fixed-point Z of F_W.

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What do we do when $H \neq 1_X$?

It suffices to consider a new safe set $W' \subseteq X$ defined by $W' = H^{-1}(W)$ and to synthesize a controller for this new set of safe states.

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Definition (Reachability game)

Let S be a system satisfying $Y = X$ and $H = 1_X$, and let $W \subseteq X$ be a set of states. The *reachability game* for system S and specification set W asks for the existence of a controller C such that for every maximal behavior $y \in \mathcal{B}(S_C) \cup \mathcal{B}^\omega(S_C)$ there exists $k \in \mathbb{N}_0$ such that $y(k) = y_k \in W$. A reachability game is said to be solvable when S_C exists.

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No longer asking for the controlled system S_C system to be nonblocking.

As in the case of safety games, it suffices to consider the case where $Y = X$ and $H = 1_X$.

The solution of reachability games is also based on a suitable operator. For any $W \subseteq X$ we can define the operator:

$$G_W : 2^X \to 2^X$$

by:

$$G_W(Z) \quad = \quad \{x \in X \mid x \in W \text{ or } \exists u \in U(x) \quad \varnothing \neq \text{Post}_u(x) \subseteq Z\}.$$

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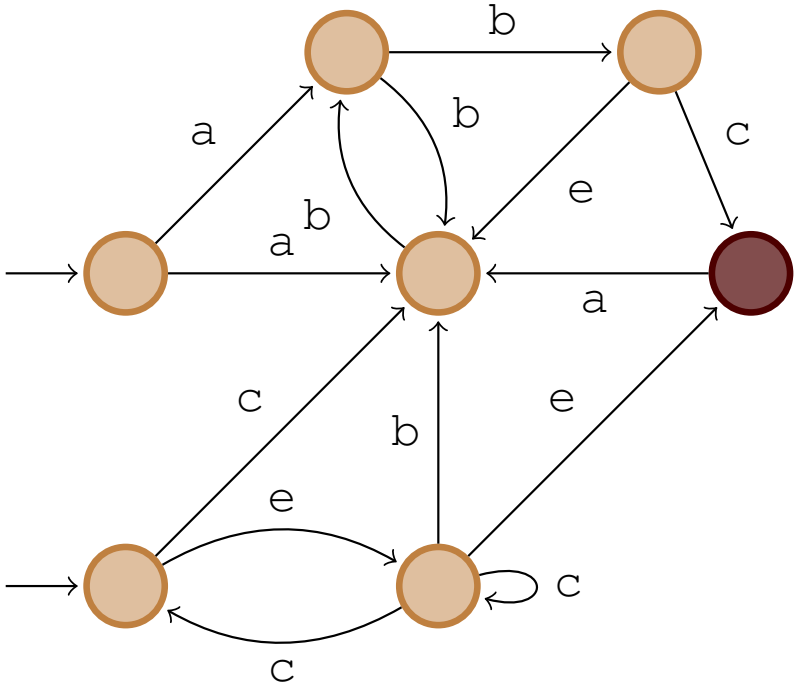
This consists of the minimum-time solution to the reachability problem. The input u belongs to $C(x)$ if there exists a k such that W can be reached from x in k steps but not in $k-1$ steps.

Consider the following finite-state system and assume that W consists of the single darker state.



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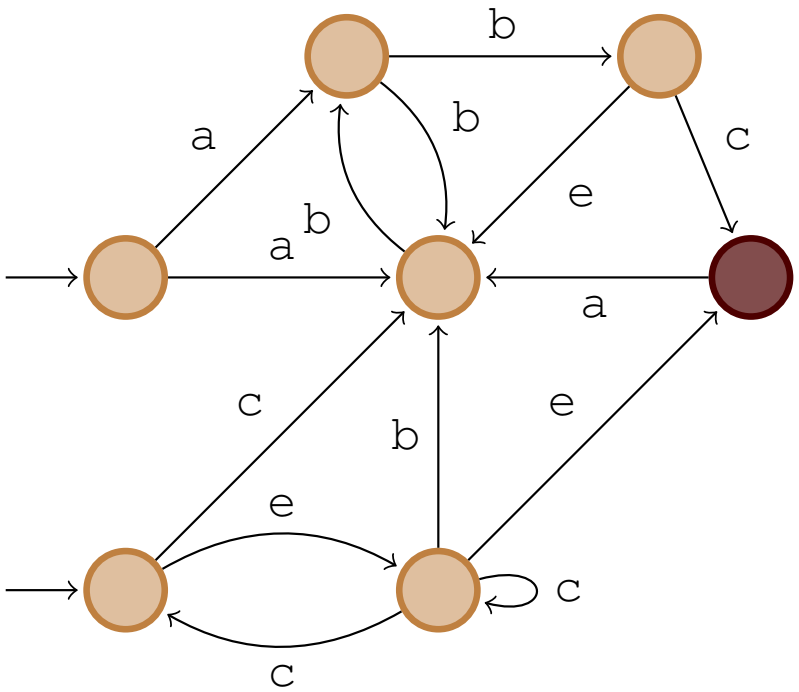

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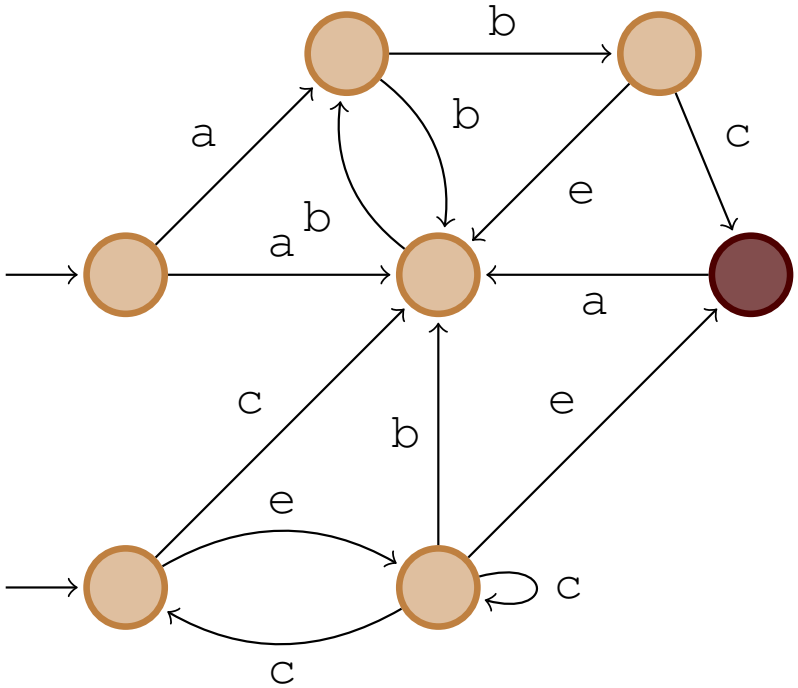

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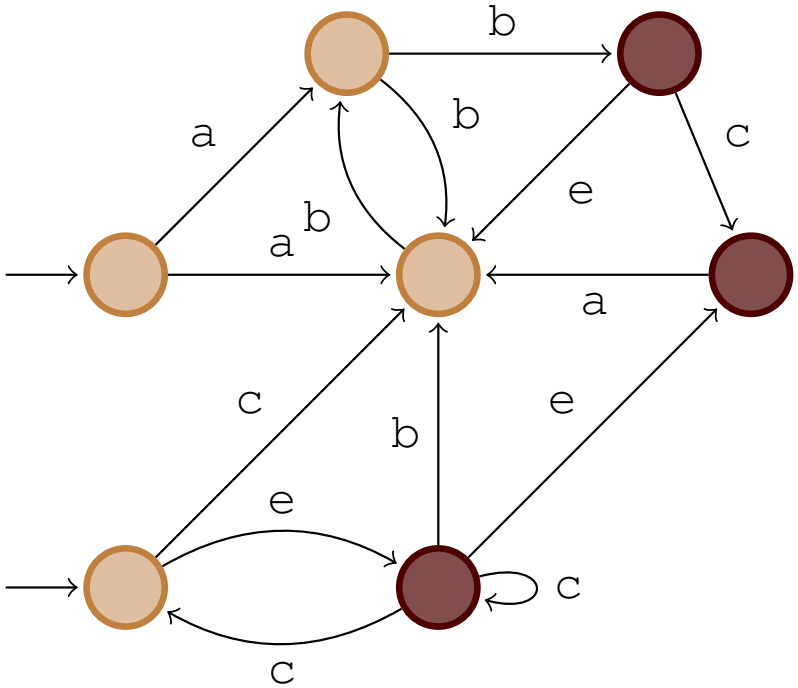

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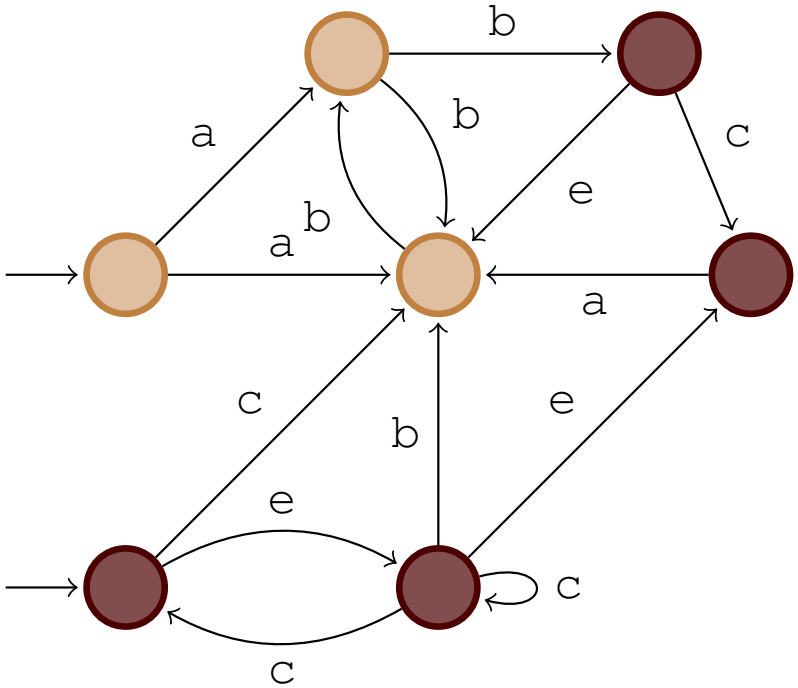

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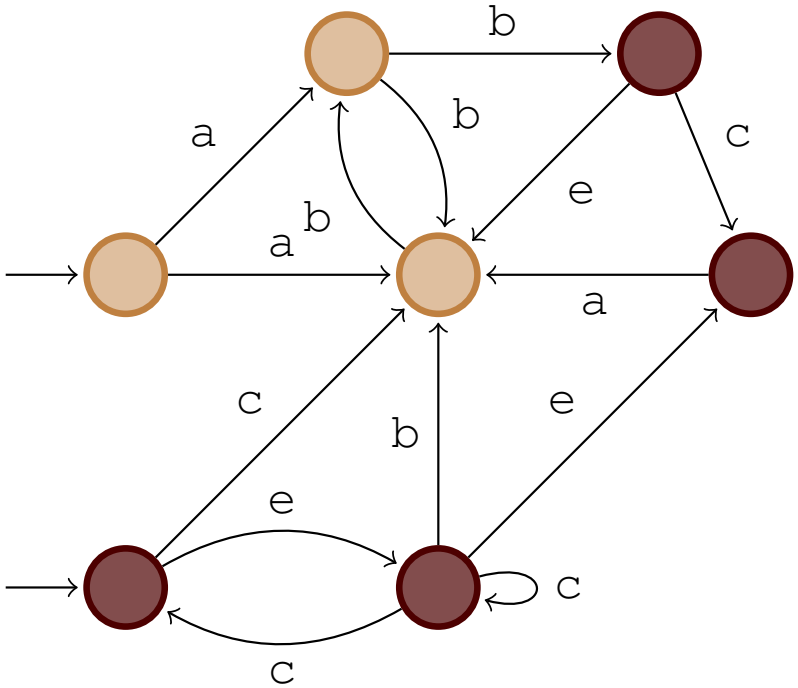

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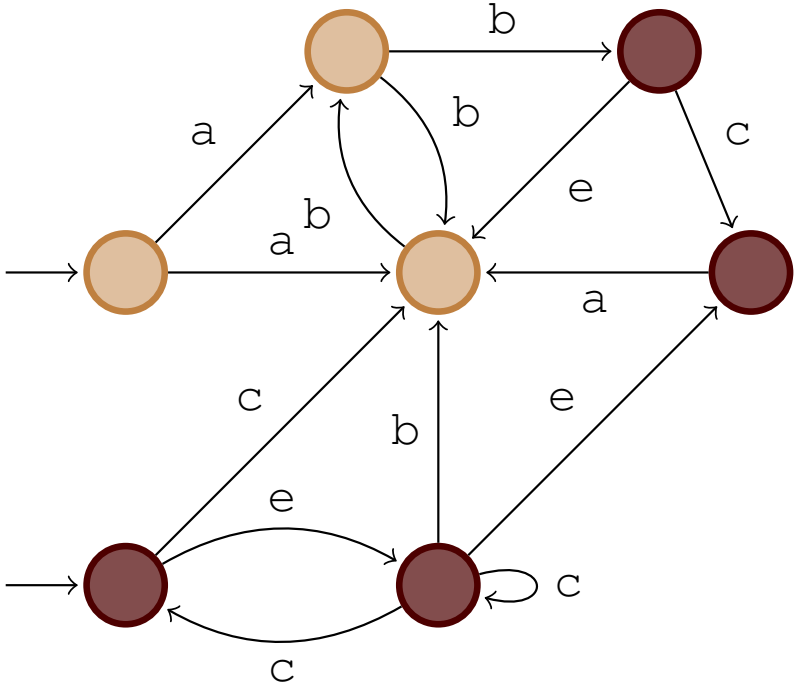
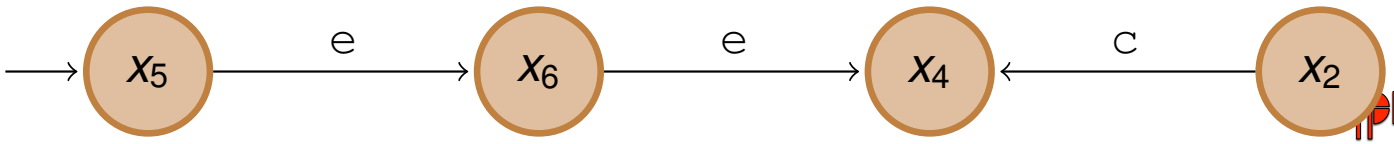
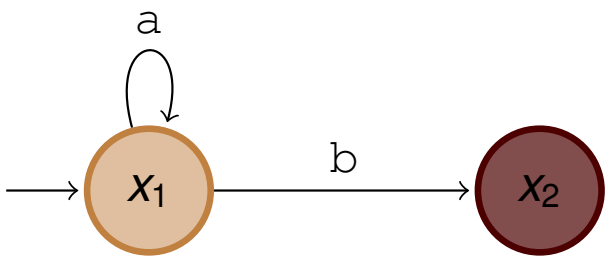


Controller:



An example to illustrate the non existence of maximal controllers:



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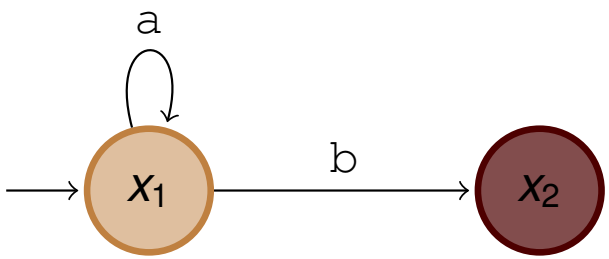

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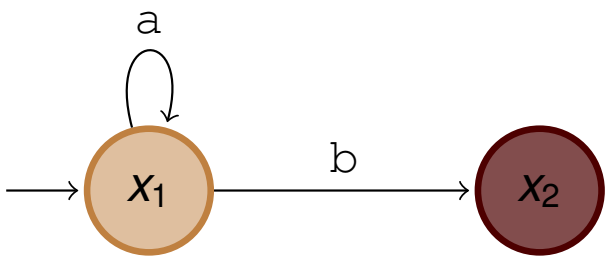

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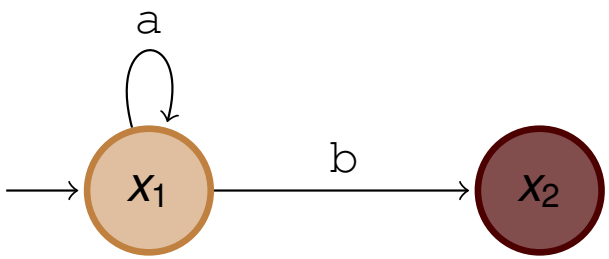

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Although these techniques cannot be directly used on infinite-state systems, they can be used when symbolic abstractions exist.